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# THE STRESS CONJUGATE TO LOGARITHMIC STRAIN

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**Abstract**—A stress **S** is said to be conjugate to a strain measure **E** if the inner product $\mathbf{S}\cdot\dot{\mathbf{E}}$ is the power per unit volume. The logarithmic strain ln **U**, with **U** the right stretch tensor, has been considered an interesting strain measure because of the relationship of its material time derivative (ln **U**)˙ with the stretching tensor **D**. In a previous article (*Int. J. Solids Structures* **22**, 1019–1032 (1986)) a formula for (ln **U**)˙ was obtained in direct notation for the cases where the principal stretches are repeated, as well as for the case where they are all distinct. Here the formula for (ln **U**)˙ and the definition of conjugate stress are used to derive an explicit, properly invariant expression for the stress conjugate to the logarithmic strain.

## 1. INTRODUCTION

The concept of conjugate stress and strain was introduced by Hill[1] as a tool with which to explore constitutive inequalities in solid mechanics. A stress **S** and a strain measure **E** are said to be a conjugate pair if $\mathbf{S}\cdot\dot{\mathbf{E}}$ represents the power per unit volume, so a necessary and sufficient condition for **S** and **E** to be a conjugate pair is that they satisfy

$$\mathbf{S}\cdot\dot{\mathbf{E}} = (\det \mathbf{U})\mathbf{T}\cdot\mathbf{D}.$$

Here **U**, **T** and **D** represent the right stretch, Cauchy stress, and stretching tensors, respectively.

The logarithmic strain ln **U** and the related ln **V**, with **V** the left stretch, have been considered useful strain measures, and have enjoyed particular attention because of the relationships of their material time derivatives to the stretching tensor **D** (e.g. see Refs [2–5]).

The problem of finding the stress conjugate to the strain ln **U** has been addressed and partially resolved by Hill[3, 6]. He obtained the components of this stress with respect to the principal axes of **U** for the case where the principal values of **U** are distinct.

In this paper we find explicit formulas for the stress conjugate to ln **U** which are expressed in direct notation and hold for repeated as well as all distinct principal stretches. The strain measure ln **V** has no conjugate stress (see Ref. [7] and the discussion closing Subsection 3.3).

Section 2 contains a brief summary of the kinematical results which will be used in the remainder of the paper. The notion of conjugate stress and strain is precisely defined in that section as well, and we discuss the conditions under which, given a strain measure, the corresponding conjugate stress is uniquely determined.

In Section 3 the stress conjugate to the logarithmic strain is obtained. The method used depends on finding a formula for **D** in terms of ln **U**. Such a formula is derived for the case of two distinct principal stretches in Subsection 3.1, and for three distinct principal stretches in Subsection 3.2. The stress conjugate to ln **U** is found by substitution of these expressions into $(\det \mathbf{U})\mathbf{T}\cdot\mathbf{D}$; this is carried out in Subsection 3.3.

A variant of the basic procedure used here to find the stress conjugate to the logarithmic strain can be successfully applied to the problem of determining the conjugate stress to any strain measure which can be written in the form

$$\mathbf{H} = \sum_{i=1}^{3} h(\lambda_i)\mathbf{e}_i \otimes \mathbf{e}_i$$

with $h$ a scalar valued strictly monotone function, and $\lambda_i$ and $\mathbf{e}_i$ a principal stretch and associated principal axis. This result is included in an article on general isotropic strain measures[8].

## 2. PRELIMINARIES†

Let $\mathbf{F}$ denote the deformation gradient at a point of a deforming body. The requirement that $\det \mathbf{F} > 0$ allows the unique polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{1}$$

where $\mathbf{U}$ and $\mathbf{V}$, the right and left stretch tensors, respectively, are positive definite symmetric, and the rotation $\mathbf{R}$ is proper orthogonal.

It is assumed that $\mathbf{F}$ is a continuously differentiable function of time. The spacial velocity gradient

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \qquad (2)‡$$

has as its symmetric part the stretching tensor $\mathbf{D}$.

The eigenvalues of $\mathbf{U}$, which are also those of $\mathbf{V}$, are termed the principal stretches and denoted by $\lambda_1$, $\lambda_2$ and $\lambda_3$. The principal invariants of $\mathbf{U}$ and $\mathbf{V}$ are

$$\begin{aligned} I &= \lambda_1 + \lambda_2 + \lambda_3 \\ II &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ III &= \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{3}$$

The Cayley–Hamilton theorem states that every tensor satisfies its own characteristic equation; e.g. $\mathbf{U}$ meets

$$\mathbf{U}^3 - I\,\mathbf{U}^2 + II\,\mathbf{U} - III\,\mathbf{1} = \mathbf{0}. \tag{4}$$

By the spectral theorem $\mathbf{U}$ has the representation

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i \tag{5}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis of eigenvectors of $\mathbf{U}$ with the eigenvector, or principal axis, $\mathbf{e}_i$ corresponding to principal stretch $\lambda_i$. The principal axes of $\mathbf{V}$, $\{\bar{\mathbf{e}}_i\}$, are related to the principal axes of $\mathbf{U}$ through

$$\bar{\mathbf{e}}_i = \mathbf{R}\mathbf{e}_i.$$

The tensor logarithm maps positive definite symmetric tensors into symmetric tensors. The logarithmic strain tensor $\ln \mathbf{U}$ is defined as

$$\ln \mathbf{U} = \sum_{i=1}^{3} \ln \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i. \tag{6}$$

Expressions for the material time derivative of the logarithmic strain were derived in Ref. [5]; the results of interest here are displayed below.

† The notation and terminology of Ref. [9] are followed.
‡ A superposed dot will be used to indicate the material time derivative.

Suppose that there are three distinct principal stretches, i.e. $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. Then

$$(\ln \mathbf{U})^{\cdot} = \mathbf{R}^T[\Phi_1\mathbf{D}+\Phi_2(\mathbf{DV}+\mathbf{VD})+\Phi_3(\mathbf{V}^2\mathbf{D}+\mathbf{DV}^2)+\Phi_4\mathbf{VDV} + \Phi_5(\mathbf{V}^2\mathbf{DV}+\mathbf{VDV}^2)+\Phi_6\mathbf{V}^2\mathbf{DV}^2]\mathbf{R} \quad (7)$$

where†

$$\begin{aligned} \Phi_1 &= 1-2III\Phi_5 - I\,III\Phi_6 \\ \Phi_2 &= II\Phi_5+\tfrac{1}{2}(I\,II-III)\Phi_6 \\ \Phi_4 &= -2\Phi_3-2I\Phi_5-(I^2-II)\Phi_6. \end{aligned} \quad (8)$$

The remaining coefficients are defined as follows:

$$\begin{aligned} \Phi_3 &= III\sum_{i=1}^{3} H_i \\ \Phi_5 &= \frac{1}{[I\,II-III]}\sum_{i=1}^{3}(\{-I\,II^2+2\phi_i[I\,II-III]-[\psi_i^2-2\phi_i]III-\phi_i^2 I\}H_i+\psi_i I G_i) \\ \Phi_6 &= \frac{1}{I\,II-III}\sum_{i=1}^{3}\{[(I\,II-III)I+II^2-2\psi_i(I\,II-III)+(\psi_i^2-2\phi_i)II+\phi_i^2]H_i+\psi_i G_i\} \end{aligned} \quad (9)‡$$

with

$$\begin{aligned} \phi_i &= \lambda_j\lambda_k \\ \psi_i &= \lambda_j+\lambda_k \\ G_i &= \frac{\ln\lambda_i}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)} \\ H_i &= \frac{\lambda_i^{-1}-(\lambda_i-\lambda_j)(G_i+G_k)-(\lambda_i-\lambda_k)(G_i+G_j)}{(\lambda_i-\lambda_j)^2(\lambda_i-\lambda_k)^2} \end{aligned} \quad (10)$$

and $i, j, k$ an even permutation of 1, 2, 3.

Suppose that there are exactly two distinct principal stretches, say§ $\lambda_1 \neq \lambda_2 = \lambda_3 =: \lambda_0$. Then

$$(\ln \mathbf{U})^{\cdot} = \mathbf{R}^T[\Theta_1\mathbf{D}+\Theta_2(\mathbf{VD}+\mathbf{DV})+\Theta_3\mathbf{VDV}]\mathbf{R} \quad (11)$$

where

$$\begin{aligned} \Theta_1 &= 1+\lambda_1\lambda_0\Theta_3 \\ \Theta_2 &= -\tfrac{1}{2}(\lambda_1+\lambda_0)\Theta_3 \\ \Theta_3 &= \frac{2(\lambda_1^2-\lambda_0^2)-4\lambda_1\lambda_0\ln(\lambda_1/\lambda_0)}{(\lambda_1+\lambda_0)(\lambda_1-\lambda_0)^3}. \end{aligned} \quad (12)$$

Suppose that there is one distinct principal stretch, i.e. $\lambda_1 = \lambda_2 = \lambda_3$. Then

† These relations are not shown explicitly in Ref. [5]; they can be proved by using the expressions for the $\Phi_i$ given there and some tedious algebra.

‡ The summation convention is not employed; i.e. summation over repeated indices is not implied.

§ All cases where it is assumed that $\lambda_1 \neq \lambda_2 = \lambda_3$ are easily generalized to $\lambda_i \neq \lambda_j = \lambda_k$, with $i, j, k$ any permutation of 1, 2, 3. The notation $A := B$ indicates an equality in which $A$ is defined by $B$.

$$(\ln \mathbf{U})^{\cdot} = \mathbf{R}^T\mathbf{D}\mathbf{R}. \tag{13}$$

For the work at hand, the following formulation of the notion of conjugate stress and strain will be most useful. *Let* $\mathbf{E}$ *be a symmetric tensor valued measure of strain. If there exists a symmetric tensor* $\mathbf{S}$ *such that*

$$\mathbf{S}\cdot\dot{\mathbf{E}} = III\,\mathbf{T}\cdot\mathbf{D} \qquad (14)\dagger$$

*for all motions, then* $\mathbf{S}$ *is the stress conjugate to* $\mathbf{E}$. As noted in the introduction, $III\,\mathbf{T}\cdot\mathbf{D}$ is the stress power per unit volume.

Note that, given a particular strain measure $\mathbf{E}$, there can be at most one corresponding conjugate stress $\mathbf{S}$ as long as $\dot{\mathbf{E}}$, taken over the set of all motions, spans the space of symmetric tensors. Also, in the present case nothing would be gained by allowing $\mathbf{S}$ to be asymmetric as the left-hand side of eqn (14) is insensitive to the skew part of $\mathbf{S}$.

Not every stress tensor has a conjugate strain associated with it. For example, because the stretching tensor $\mathbf{D}$ is not a material time derivative of a strain measure, the Cauchy stress is not part of a conjugate pair (see e.g. Ref. [7] or Ref. [1]). Similarly, there are strains which do not have a conjugate stress. One such strain measure is the logarithm of the left stretch, $\ln \mathbf{V}$, as will be shown at the end of Subsection 3.3.

It will be shown that there exists a unique symmetric stress conjugate to $\ln \mathbf{U}$, which we will denote by $\mathbf{T}^{(0)}$, and with the aid of eqn (14) an explicit formula will be derived for that stress.

Before proceeding we note that $(\ln \mathbf{U})^{\cdot}$, taken over the set of all motions, spans the space of symmetric tensors. From eqns (7), (11) and (13) it is evident that $(\ln \mathbf{U})^{\cdot}$ is an isotropic function of $\mathbf{D}$ and $\mathbf{V}$, linear in $\mathbf{D}$. In fact, as calculated in Ref. [5]

$$[(\ln \mathbf{U})^{\cdot}]_{ij} = \begin{cases} D_{ii}, & i=j \\ D_{ij}, & i\neq j, \lambda_i=\lambda_j \\ \dfrac{2\lambda_i\lambda_j \ln(\lambda_i/\lambda_j)}{\lambda_i^2-\lambda_j^2}D_{ij}, & i\neq j, \lambda_i\neq\lambda_j \end{cases} \tag{15}$$

where the components are taken with respect to the principal axes of $\mathbf{U}$ and $\mathbf{V}$ on the left- and right-hand sides, respectively. Since the stretching tensor $\mathbf{D}$, when considered over all motions, spans the space of symmetric tensors,‡ it is clear from eqn (15) that so, also, does $\ln \mathbf{U}$.

## 3. THE STRESS CONJUGATE TO ln U

In this section the stress conjugate to $\ln \mathbf{U}$ will be determined by the following procedure. First, the expressions giving $(\ln \mathbf{U})^{\cdot}$ in terms of $\mathbf{D}$ displayed in Section 2 will be inverted to give equations for $\mathbf{D}$ written in terms of $(\ln \mathbf{U})^{\cdot}$. It is necessary that the number of distinct principal stretches be specified; in the case of one distinct principal stretch the inversion is immediate, the case where there are exactly two distinct principal stretches is dealt with in Subsection 3.1, and the case of three distinct principal stretches is addressed in Subsection 3.2. These expressions for $\mathbf{D}$ will then be substituted into $III\,\mathbf{T}\cdot\mathbf{D}$ to yield explicit formulas for the stress $\mathbf{T}^{(0)}$ conjugate to the logarithmic strain; these are displayed in Subsection 3.3.

### 3.1. *Two distinct principal stretches*

Throughout this subsection it will be assumed that there are exactly two distinct principal stretches, say

† The inner product of two tensors $\mathbf{A}$ and $\mathbf{B}$ will be denoted by $\mathbf{A}\cdot\mathbf{B} := \operatorname{tr}(\mathbf{A}^T\mathbf{B})$.

‡ It suffices to consider motions of the form $\mathbf{x}(\mathbf{p}, t) = t\mathbf{A}(\mathbf{p}-\mathbf{p}_0)$, where $\mathbf{A}$ is an arbitrary constant symmetric tensor and $\mathbf{p}_0$ is an arbitrary fixed point; for then $\mathbf{D} = \mathbf{A}$.

$$\lambda_1 \neq \lambda_2 = \lambda_3 =: \lambda_0. \tag{16}$$

In this case $(\ln \mathbf{U})^{\cdot}$ is given in terms of $\mathbf{D}$, $\mathbf{V}$, $\lambda_1$ and $\lambda_0$ by eqn (11). Our purpose here is to obtain an expression for $\mathbf{D}$ in terms of $(\ln \mathbf{U})^{\cdot}$ valid for this case.

For convenience let

$$\mathbf{P} := \mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^T \tag{17}$$

then eqn (11) can be written as

$$\mathbf{P} = \Theta_1\mathbf{D}+\Theta_2(\mathbf{VD}+\mathbf{DV})+\Theta_3\mathbf{VDV} \tag{18}$$

with the $\Theta_i$ given by eqn (12). With respect to the principal axes of $\mathbf{V}$ the components of $\mathbf{P}$ are

$$P_{ij} = D_{ij}[\Theta_1+\Theta_2(\lambda_i+\lambda_j)+\Theta_3\lambda_i\lambda_j] \tag{19}$$

where $i$ and $j$ range over 1, 2, 3. With the aid of eqn (12) the term in square brackets is easily calculated:

$$\begin{aligned}
&\Theta_1+2\Theta_2\lambda_i+\Theta_3\lambda_i^2 = 1, && i=j\\
&\Theta_1+\Theta_2(\lambda_i+\lambda_j)+\Theta_3\lambda_i\lambda_j = \frac{2\lambda_1\lambda_0 \ln(\lambda_1/\lambda_0)}{\lambda_1^2-\lambda_0^2}, && i\neq j, \lambda_i \neq \lambda_j\\
&\Theta_1+\Theta_2(\lambda_i+\lambda_j)+\Theta_3\lambda_i\lambda_j = 1, && i\neq j, \lambda_i = \lambda_j.
\end{aligned} \tag{20}$$

In view of eqn (16), the right-hand side of eqn $(20)_2$ is non-zero; thus eqn (19) may be inverted to give

$$D_{ij} = \begin{cases} P_{ii}, & i=j\\ P_{ij}\dfrac{\lambda_1^2-\lambda_0^2}{2\lambda_1\lambda_0 \ln(\lambda_1/\lambda_0)}, & i\neq j, \lambda_i\neq\lambda_j\\ P_{ij}, & i\neq j, \lambda_i=\lambda_j. \end{cases} \tag{21}$$

We now seek scalars $\Psi_i$ such that $\mathbf{D}$ may be written as

$$\mathbf{D} = \Psi_1\mathbf{P}+\Psi_2(\mathbf{VP}+\mathbf{PV})+\Psi_3\mathbf{VPV}. \tag{22}$$

In component form with respect to the principal axes of $\mathbf{V}$, eqn (22) states that

$$D_{ij} = P_{ij}[\Psi_1+\Psi_2(\lambda_i+\lambda_j)+\Psi_3\lambda_i\lambda_j]. \tag{23}$$

By substituting the values for $D_{ij}$ given by eqns (21) into the left-hand side of eqn (23), we obtain

$$\begin{aligned}
\Psi_1+2\Psi_2\lambda_1+\Psi_3\lambda_1^2 &= 1\\
\Psi_1+2\Psi_2\lambda_0+\Psi_3\lambda_0^2 &= 1\\
\Psi_1+\Psi_2(\lambda_1+\lambda_0)+\Psi_3\lambda_1\lambda_0 &= \frac{\lambda_1^2+\lambda_0^2}{2\lambda_1\lambda_0\ln(\lambda_1/\lambda_0)}.
\end{aligned} \tag{24}$$

Treating the $\Psi_i$ as unknowns for which to solve, one can write system (24) as the matrix equation

$$\begin{bmatrix} 1 & 2\lambda_1 & \lambda_1^2 \\ 1 & 2\lambda_0 & \lambda_0^2 \\ 1 & (\lambda_1+\lambda_0) & \lambda_1\lambda_0 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \dfrac{\lambda_1^2+\lambda_0^2}{2\lambda_1\lambda_0 \ln(\lambda_1/\lambda_0)} \end{bmatrix}. \tag{25}$$

The determinant of the coefficient matrix is

$$(\lambda_1-\lambda_0)^3 \neq 0$$

by eqn (16). Thus eqn (25) has a unique solution, which by Cramer's rule is

$$\begin{aligned} \Psi_1 &= \frac{-(\lambda_1-\lambda_0)^2+(\lambda_1^2+\lambda_0^2)\ln(\lambda_1/\lambda_0)}{(\lambda_1-\lambda_0)^2 \ln(\lambda_1/\lambda_0)} \\ \Psi_2 &= \frac{(\lambda_1+\lambda_0)[(\lambda_1^2-\lambda_0^2)-2\lambda_1\lambda_0\ln(\lambda_1/\lambda_0)]}{2\lambda_1\lambda_0(\lambda_1-\lambda_0)^2\ln(\lambda_1/\lambda_0)} \\ \Psi_3 &= \frac{-(\lambda_1^2-\lambda_0^2)+2\lambda_1\lambda_0\ln(\lambda_1/\lambda_0)}{\lambda_1\lambda_0(\lambda_1-\lambda_0)^2\ln(\lambda_1/\lambda_0)}. \end{aligned} \tag{26}$$

Note that the coefficients $\Psi_i$ satisfy

$$\begin{aligned} \Psi_1 &= 1+\lambda_1\lambda_0\Psi_3 \\ \Psi_2 &= -\tfrac{1}{2}(\lambda_1+\lambda_0)\Psi_3. \end{aligned} \tag{27}$$

It is interesting that these parallel the relations among the coefficients $\Theta_i$ of eqn (18) (see eqn (12)).

We have established that *if* **D** *admits the representation* (22), *the scalars* $\Psi_i$ *are given by eqns* (26). *Conversely, if the* $\Psi_i$ *are defined by eqns* (26), *then eqns* (24) *and therefore eqn* (22) *holds. So the stretching tensor* **D** *can be written in terms of* (ln **U**)$^{\cdot}$ *and* **U** *in the form*

$$\mathbf{D} = \mathbf{R}\{\Psi_1(\ln\mathbf{U})^{\cdot}+\Psi_2[\mathbf{U}(\ln\mathbf{U})^{\cdot}+(\ln\mathbf{U})^{\cdot}\mathbf{U}]+\Psi_3\mathbf{U}(\ln\mathbf{U})^{\cdot}\mathbf{U}\}\mathbf{R}^T \tag{28}$$

*with the* $\Psi_i$ *uniquely given by eqns* (26) *as functions of the two distinct principal stretches.* Equation (1) was employed to write eqn (28) in terms of **U** rather than **V**.

### 3.2. *Three distinct principal stretches*

The purpose of this subsection is to obtain an expression for **D** in terms of (ln **U**)$^{\cdot}$ which is valid when there are three distinct principal stretches, i.e. when

$$\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1. \tag{29}$$

The same basic method as was used in Subsection 3.1 will be used here.

We again employ the definition

$$\mathbf{P} = \mathbf{R}(\ln\mathbf{U})^{\cdot}\mathbf{R}^T \tag{30}$$

and write eqn (7) as

$$\mathbf{P} = \Phi_1\mathbf{D}+\Phi_2(\mathbf{DV}+\mathbf{VD})+\Phi_3(\mathbf{V}^2\mathbf{D}+\mathbf{DV}^2)+\Phi_4\mathbf{VDV}+\Phi_5(\mathbf{V}^2\mathbf{DV}+\mathbf{VDV}^2)+\Phi_6\mathbf{V}^2\mathbf{DV}^2 \tag{31}$$

where the coefficients $\Phi_i$ are given by eqns (8) and (9).

With respect to the principal axes of **V** the components of **P** are

$$P_{ij} = D_{ij}[\Phi_1+\Phi_2(\lambda_i+\lambda_j)+\Phi_3(\lambda_i^2+\lambda_j^2)+\Phi_4\lambda_i\lambda_j+\Phi_5\lambda_i\lambda_j(\lambda_i+\lambda_j)+\Phi_6\lambda_i^2\lambda_j^2] \quad (32)$$

where $i, j$ range over 1, 2, 3. With the aid of eqns (8) and (9), a fairly involved calculation shows that the term in brackets reduces to

$$\Phi_1+\Phi_2(\lambda_i+\lambda_j)+\Phi_3(\lambda_i^2+\lambda_j^2)+\Phi_4\lambda_i\lambda_j+\Phi_5\lambda_i\lambda_j(\lambda_i+\lambda_j)+\Phi_6\lambda_i^2\lambda_j^2 = \begin{cases} 1, & i=j \\ \dfrac{2\lambda_i\lambda_j \ln(\lambda_i/\lambda_j)}{\lambda_i^2-\lambda_j^2}, & i\neq j \end{cases} \quad (33)$$

Substituting eqn (33) into eqn (32), we obtain

$$P_{ij} = \begin{cases} D_{ii}, & i=j \\ D_{ij}\dfrac{2\lambda_i\lambda_j \ln(\lambda_i/\lambda_j)}{\lambda_i^2-\lambda_j^2}, & i\neq j \end{cases}. \quad (34)\dagger$$

Recall that the principal stretches are all distinct here (see condition (29)), so the coefficient of $D_{ij}$ does not vanish. Thus eqn (34) may be inverted:

$$D_{ij} = \begin{cases} P_{ii}, & i=j \\ P_{ij}\dfrac{\lambda_i^2-\lambda_j^2}{2\lambda_i\lambda_j \ln(\lambda_i/\lambda_j)}, & i\neq j \end{cases}. \quad (35)$$

We next establish that unique coefficients $\Lambda_i$ can be found such that $\mathbf{D}$ can be written in terms of $\mathbf{P} = \mathbf{R}(\ln \mathbf{U})^{\cdot}\mathbf{R}^T$as

$$\mathbf{D} = \Lambda_1\mathbf{P}+\Lambda_2(\mathbf{VP}+\mathbf{PV})+\Lambda_3(\mathbf{V}^2\mathbf{P}+\mathbf{PV}^2)+\Lambda_4\mathbf{VPV}+\Lambda_5(\mathbf{V}^2\mathbf{PV}+\mathbf{VPV}^2)+\Lambda_6\mathbf{V}^2\mathbf{PV}^2. \quad (36)$$

The $\Lambda_i$ will be scalar functions of the principal stretches.

With respect to the principal axes of $\mathbf{V}$ the component form of eqn (36) is

$$D_{ij} = P_{ij}[\Lambda_1+\Lambda_2(\lambda_i+\lambda_j)+\Lambda_3(\lambda_i^2+\lambda_j^2)+\Lambda_4\lambda_i\lambda_j+\Lambda_5\lambda_i\lambda_j(\lambda_i+\lambda_j)+\Lambda_6\lambda_i^2\lambda_j^2]. \quad (37)$$

The left-hand members of eqns (35) and (37) are identical so, by setting the right-hand sides equal to each other, we find that if such $\Lambda_i$ exist they must satisfy

$$\Lambda_1+\Lambda_2(\lambda_i+\lambda_j)+\Lambda_3(\lambda_i^2+\lambda_j^2)+\Lambda_4\lambda_i\lambda_j+\Lambda_5\lambda_i\lambda_j(\lambda_i+\lambda_j)+\Lambda_6\lambda_i^2\lambda_j^2 = \begin{cases} 1, & i=j \\ \dfrac{\lambda_i^2-\lambda_j^2}{2\lambda_i\lambda_j \ln(\lambda_i/\lambda_j)}, & i\neq j \end{cases}. \quad (38)$$

As the indices range over 1, 2, 3 eqn (38) yields a system of six equations which can be written in matrix form as

$$[M][\Lambda] = [N] \quad (39)$$

where $[\Lambda]$ represents the column matrix with components $\Lambda_i$

† This result was established by a different method by Hill[3].

$$[M] = \begin{bmatrix} 1 & 2\lambda_1 & 2\lambda_1^2 & \lambda_1^2 & 2\lambda_1^3 & \lambda_1^4 \\ 1 & 2\lambda_2 & 2\lambda_2^2 & \lambda_2^2 & 2\lambda_2^3 & \lambda_2^4 \\ 1 & 2\lambda_3 & 2\lambda_3^2 & \lambda_3^2 & 2\lambda_3^3 & \lambda_3^4 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 & \lambda_1\lambda_2 & \lambda_1\lambda_2(\lambda_1+\lambda_2) & \lambda_1^2\lambda_2^2 \\ 1 & \lambda_1+\lambda_3 & \lambda_1^2+\lambda_3^2 & \lambda_1\lambda_3 & \lambda_1\lambda_3(\lambda_1+\lambda_3) & \lambda_1^2\lambda_3^2 \\ 1 & \lambda_2+\lambda_3 & \lambda_2^2+\lambda_3^2 & \lambda_2\lambda_3 & \lambda_2\lambda_3(\lambda_2+\lambda_3) & \lambda_2^2\lambda_3^2 \end{bmatrix}$$

and

$$[N] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \dfrac{\lambda_1^2-\lambda_2^2}{2\lambda_1\lambda_2 \ln(\lambda_1/\lambda_2)} \\ \dfrac{\lambda_1^2-\lambda_3^2}{2\lambda_1\lambda_3 \ln(\lambda_1/\lambda_3)} \\ \dfrac{\lambda_2^2-\lambda_3^2}{2\lambda_2\lambda_3 \ln(\lambda_2/\lambda_3)} \end{bmatrix}.$$

With $R_i$ representing the $i$th row of $[M]$, we form a matrix $[\bar{M}]$ the rows of which are $R_1$, $R_2-R_1$, $R_3-R_1$, $-2R_4+R_1+R_2$, $-2R_5+R_1+R_3$ and $-2R_6+R_2+R_3$

$$[\bar{M}] = \begin{bmatrix} 1 & 2\lambda_1 & 2\lambda_1^2 & \lambda_1^2 & 2\lambda_1^3 & \lambda_1^4 \\ 0 & 2(\lambda_2-\lambda_1) & 2(\lambda_2^2-\lambda_1^2) & \lambda_2^2-\lambda_1^2 & 2(\lambda_2^3-\lambda_1^3) & \lambda_2^4-\lambda_1^4 \\ 0 & 2(\lambda_3-\lambda_1) & 2(\lambda_3^2-\lambda_1^2) & \lambda_3^2-\lambda_1^2 & 2(\lambda_3^3-\lambda_1^3) & \lambda_3^4-\lambda_1^4 \\ 0 & 0 & 0 & (\lambda_1-\lambda_2)^2 & (\lambda_1-\lambda_2)^2(\lambda_1+\lambda_2) & (\lambda_1^2-\lambda_2^2)^2 \\ 0 & 0 & 0 & (\lambda_1-\lambda_3)^2 & (\lambda_1-\lambda_3)^2(\lambda_1+\lambda_3) & (\lambda_1^2-\lambda_3^2)^2 \\ 0 & 0 & 0 & (\lambda_2-\lambda_3)^2 & (\lambda_2-\lambda_3)^2(\lambda_2+\lambda_3) & (\lambda_2^2-\lambda_3^2)^2 \end{bmatrix}. \quad (40)$$

The determinant of $[M]$ is now easily calculated to be

$$\det [M] = \det [\bar{M}] = 4(\lambda_1-\lambda_2)^4(\lambda_2-\lambda_3)^4(\lambda_3-\lambda_1)^4.$$

Condition (29) implies that $\det [M] \neq 0$, so eqn (39) possesses a unique solution for the $\Lambda_i$.

Rather than solving eqn (39) directly for all six of the coefficients $\Lambda_i$, we will conjecture that the coefficients satisfy the relations

$$\begin{aligned} \Lambda_1 &= 1-2III\Lambda_5 - I\,III\Lambda_6 \\ \Lambda_2 &= II\Lambda_5 + \tfrac{1}{2}(I\,II - III)\Lambda_6 \\ \Lambda_4 &= -2\Lambda_3 - 2I\Lambda_5 - (I^2 - II)\Lambda_6 \end{aligned} \quad (41)$$

and solve for $\Lambda_3$, $\Lambda_5$, $\Lambda_6$. This conjecture is motivated by the observation that, in the case of two distinct principal stretches, the coefficients $\Theta_i$ in eqn (18) and $\Psi_i$ in eqn (28) satisfy similar relations (see the remark following eqns (27)). The relationships among the $\Lambda_i$ in eqns (41) parallel conditions (8) for the coefficients $\Phi_i$ of eqn (31).

With eqns (41), eqn $(38)_1$ is automatically satisfied, and eqn $(38)_2$ yields a system of three equations which can be written in matrix form as

$$[M'] = [\Lambda'][N'] \tag{42}$$

with

$$[M'] = \begin{bmatrix} (\lambda_1-\lambda_2)^2 & \lambda_3(\lambda_1-\lambda_2)^2 & \frac{1}{2}[\lambda_3^2+\lambda_3(\lambda_1+\lambda_2)-\lambda_1\lambda_2](\lambda_1-\lambda_2)^2 \\ (\lambda_1-\lambda_3)^2 & \lambda_2(\lambda_1-\lambda_3)^2 & \frac{1}{2}[\lambda_2^2+\lambda_2(\lambda_1+\lambda_3)-\lambda_1\lambda_3](\lambda_1-\lambda_3)^2 \\ (\lambda_2-\lambda_2)^3 & \lambda_1(\lambda_2-\lambda_3)^2 & \frac{1}{2}[\lambda_1^2+\lambda_1(\lambda_2+\lambda_3)-\lambda_2\lambda_3](\lambda_2-\lambda_3)^2 \end{bmatrix}$$

$$[\Lambda'] = \begin{bmatrix} \Lambda_3 \\ \Lambda_5 \\ \Lambda_6 \end{bmatrix}$$

and

$$[N'] = \begin{bmatrix} \dfrac{\lambda_1^2-\lambda_2^2}{2\lambda_1\lambda_2 \ln(\lambda_1/\lambda_2)} - 1 \\ \dfrac{\lambda_1^2-\lambda_3^2}{2\lambda_1\lambda_3 \ln(\lambda_1/\lambda_3)} - 1 \\ \dfrac{\lambda_2^2-\lambda_3^2}{2\lambda_2\lambda_3 \ln(\lambda_2/\lambda_3)} - 1 \end{bmatrix}.$$

The determinant of $[M']$ is

$$\tfrac{1}{2}(\lambda_1-\lambda_2)^3(\lambda_2-\lambda_3)^3(\lambda_3-\lambda_1)^3$$

which by condition (29) does not vanish. Thus, eqn (42) has a unique solution, and calculation by Cramer's rule gives the following result. Let

$$\phi_i = \lambda_j\lambda_k$$
$$\psi_i = \lambda_j+\lambda_k$$

as before (see eqns (10)), and define

$$\begin{aligned} \mu_i &= \ln(\lambda_j/\lambda_k) \\ v_i &= \ln(\lambda_i/\lambda_j)\ln(\lambda_k/\lambda_i) \\ \Delta &= (\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1) \\ \Gamma &= 2\lambda_1\lambda_2\lambda_3 \ln(\lambda_1/\lambda_2)\ln(\lambda_2/\lambda_3)\ln(\lambda_3/\lambda_1) \end{aligned} \tag{43}$$

then

$$\begin{aligned} \Lambda_3 &= \frac{1}{\Delta^2}(4II^2-3I\,III-I^2\,II)+\frac{1}{\Gamma\Delta}\sum_{i=1}^{3}(2III\lambda_i v_i-\phi_i^2\mu_i^2) \\ \Lambda_5 &= \frac{1}{\Delta^2}(7I\,II-2I^3-9III)-\frac{1}{\Gamma\Delta}\sum_{i=1}^{3}(2\lambda_i+\psi_i)\lambda_i\psi_i v_i \\ \Lambda_6 &= \frac{2}{\Delta^2}(3II-I^2)-\frac{2}{\Delta}\sum_{i=1}^{3}\phi_i\mu_i^2. \end{aligned} \tag{44}$$

By the uniqueness of the solution of eqn (39) we have established that *the stretching tensor* **D** *can be expressed in terms of* (ln **U**)$^{\cdot}$ *and* **U** *in the form*

$$\mathbf{D} = \mathbf{R}\{\Lambda_1(\ln \mathbf{U})^{\cdot} + \Lambda_2[\mathbf{U}(\ln \mathbf{U})^{\cdot} + (\ln \mathbf{U})^{\cdot}\mathbf{U}] + \Lambda_3[\mathbf{U}^2(\ln \mathbf{U})^{\cdot} + (\ln \mathbf{U})^{\cdot}\mathbf{U}^2] + \Lambda_4\mathbf{U}(\ln \mathbf{U})^{\cdot}\mathbf{U}$$
$$+ \Lambda_5[\mathbf{U}^2(\ln \mathbf{U})^{\cdot}\mathbf{U} + \mathbf{U}(\ln \mathbf{U})^{\cdot}\mathbf{U}^2] + \Lambda_6\mathbf{U}^2(\ln \mathbf{U})^{\cdot}\mathbf{U}^2\}\mathbf{R}^T \quad (45)$$

*with the* $\Lambda_i$ *uniquely given by eqns* (41) *and* (44) *as functions of the three distinct principal stretches.* Equation (1) was used to write the above expression in terms of **U** rather than **V**.

### 3.3. *The stress* $\mathbf{T}^{(0)}$

Equations (45), (28) and (13) provide formulas for **D** in terms of $(\ln \mathbf{U})^{\cdot}$ in the case where there are three, two and one distinct principal stretches, respectively. In this subsection we will substitute these formulas for **D** into $III\,\mathbf{T}\cdot\mathbf{D}$ and find, upon comparison with eqn (14), a formula for the stress conjugate to the logarithmic strain valid for each case.

Suppose there are three distinct principal stretches. Using eqn (45) we calculate that

$$III\,\mathbf{T}\cdot\mathbf{D} = III\,\mathbf{R}^T\{\Lambda_1\mathbf{T} + \Lambda_2(\mathbf{VT}+\mathbf{TV}) + \Lambda_3(\mathbf{V}^2\mathbf{T}+\mathbf{TV}^2) + \Lambda_4\mathbf{VTV} + \Lambda_5(\mathbf{V}^2\mathbf{TV}+\mathbf{VTV}^2)$$
$$+ \Lambda_6\mathbf{V}^2\mathbf{TV}^2\}\mathbf{R}\cdot(\ln \mathbf{U})^{\cdot}. \quad (46)$$

Suppose there are exactly two distinct principal stretches. Then eqn (28) holds, so

$$III\,\mathbf{T}\cdot\mathbf{D} = III\,\mathbf{R}^T[\Psi_1\mathbf{T} + \Psi_2(\mathbf{VT}+\mathbf{TV}) + \Psi_3\mathbf{VTV}]\mathbf{R}\cdot(\ln \mathbf{U})^{\cdot}. \quad (47)$$

Suppose there is only one distinct principal stretch. Here eqn (13) applies, and

$$III\,\mathbf{T}\cdot\mathbf{D} = III\,\mathbf{R}^T\mathbf{TR}\cdot(\ln \mathbf{U})^{\cdot}. \quad (48)$$

On comparing each of eqns (46)–(48) with eqn (14), we obtain an expression for the stress conjugate to ln **U** which is valid for the case of three, two and one distinct principal stretches, respectively. The results can be gathered as follows. *The stress* $\mathbf{T}^{(0)}$ *conjugate to the logarithmic strain* ln **U** *is given by*

$$\mathbf{T}^{(0)} = \begin{cases} III\,\mathbf{R}^T[\Lambda_1\mathbf{T} + \Lambda_2(\mathbf{TV}+\mathbf{VT}) + \Lambda_3(\mathbf{V}^2\mathbf{T}+\mathbf{TV}^2) + \Lambda_4\mathbf{VTV} + \Lambda_5(\mathbf{V}^2\mathbf{TV}+\mathbf{VTV}^2) \\ \qquad + \Lambda_6\mathbf{V}^2\mathbf{TV}^2]\mathbf{R}, \qquad \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1 \\ III\,\mathbf{R}^T[\Psi_1\mathbf{T} + \Psi_2(\mathbf{VT}+\mathbf{TV}) + \Psi_3\mathbf{VTV}]\mathbf{R}, \qquad \lambda_1 \neq \lambda_2 = \lambda_3 \\ III\,\mathbf{R}^T\mathbf{TR}, \qquad \lambda_1 = \lambda_2 = \lambda_3 \end{cases} \quad (49)$$

*where the* $\Lambda_i$ *are given by eqns* (41) *and* (44) *and the* $\Psi_i$ *are displayed in eqns* (26).

The results obtained so far are independent of material characteristics, and therefore hold regardless of the constitutive equation. If, however, the material is isotropic elastic, the rotated Cauchy stress $\mathbf{R}^T\mathbf{TR}$ is an isotropic function of **U**, so it has the form

$$\mathbf{R}^T\mathbf{TR} = \alpha_0\mathbf{1} + \alpha_1\mathbf{U} + \alpha_2\mathbf{U}^2 =: \mathbf{K}(\mathbf{U})$$

with the $\alpha_i$ scalar valued functions of the principal invariants of **U**. Here the rotated Cauchy stress commutes with **U**, and use of eqn (45) with the Cayley–Hamilton theorem gives

$$\mathbf{T}\cdot\mathbf{D} = \{[\Lambda_1 + 2III\Lambda_5 + I\,III\Lambda_6]\mathbf{1} + [2\Lambda_2 - 2II\Lambda_5 - (I\,II - III)\Lambda_6]\mathbf{U}$$
$$+ [2\Lambda_3 + \Lambda_4 + 2I\Lambda_5 + (I^2 - II)\Lambda_6]\mathbf{U}^2\}\mathbf{R}^T\mathbf{K}(\mathbf{U})\mathbf{R}\cdot(\ln \mathbf{U})^{\cdot} \quad (50)$$

for three distinct principal stretches. By applying eqns (41) to the terms in square brackets, we can reduce eqn (50) to

$$\mathbf{T}\cdot\mathbf{D} = \mathbf{R}^T\mathbf{K}(\mathbf{U})\mathbf{R}\cdot(\ln\mathbf{U})^{\cdot} = \mathbf{R}^T\mathbf{T}\mathbf{R}\cdot(\ln\mathbf{U})^{\cdot}.$$

A similar result holds for the case of two distinct principal stretches. For one distinct principal stretch eqn $(49)_3$ immediately gives $\mathbf{T}\cdot\mathbf{D} = \mathbf{R}^T\mathbf{T}\mathbf{R}\cdot(\ln\mathbf{U})^{\cdot}$. Thus we directly obtain the well-known result (see Ref. [1]) that, for an isotropic elastic material, the stress conjugate to logarithmic strain is the rotated Cauchy stress multiplied by *III*.

Finally, we return to the assertion that the logarithm of the left stretch has no conjugate stress. The proof presented here is a variation of one by Ogden[7]. Since

$$\ln\mathbf{V} = \mathbf{R}(\ln\mathbf{U})\mathbf{R}^T \tag{51}$$

where $\mathbf{R}$ is the rotation, the material time derivative of $\ln\mathbf{V}$ is related to that of $\ln\mathbf{U}$ through

$$(\ln\mathbf{V})^{\cdot} = \mathbf{R}(\ln\mathbf{U})^{\cdot}\mathbf{R}^T + \dot{\mathbf{R}}\mathbf{R}^T(\ln\mathbf{V}) - (\ln\mathbf{V})\dot{\mathbf{R}}\mathbf{R}^T. \tag{52}$$

Suppose that $\ln\mathbf{V}$ has conjugate stress $\mathbf{T}^{[V]}$. Then eqn (14) requires that

$$\mathbf{T}^{[V]}\cdot(\ln\mathbf{V})^{\cdot} = III\,\mathbf{T}\cdot\mathbf{D} = \mathbf{T}^{(0)}\cdot(\ln\mathbf{U})^{\cdot}.$$

Incorporation of eqns (51) and (52) yields

$$(\mathbf{T}^{(0)} - \mathbf{R}^T\mathbf{T}^{[V]}\mathbf{R})\cdot(\ln\mathbf{U})^{\cdot} = \mathbf{R}^T\mathbf{T}^{[V]}\mathbf{R}\cdot((\ln\mathbf{U})\mathbf{R}^T\dot{\mathbf{R}} - \mathbf{R}^T\dot{\mathbf{R}}(\ln\mathbf{U})). \tag{53}$$

Recall that a conjugate pair must meet eqn (14) for all motions; thus we may consider a motion with $\dot{\mathbf{R}} = \mathbf{0}$ and $\mathbf{R} = \bar{\mathbf{R}}$, where $\bar{\mathbf{R}}$ is an arbitrary proper orthogonal tensor. Then eqn (53) implies

$$\mathbf{T}^{[V]} = \bar{\mathbf{R}}\mathbf{T}^{(0)}\bar{\mathbf{R}}^T. \tag{54}$$

As $\bar{\mathbf{R}}$ is arbitrary, eqn (54) holds for any rotation, i.e. for all motions. Consequently the right-hand member of eqn (53) must vanish. By eqn (54) this requirement may be rearranged to

$$[\mathbf{T}^{(0)}(\ln\mathbf{U}) - (\ln\mathbf{U})\mathbf{T}^{(0)}]\cdot\mathbf{R}^T\dot{\mathbf{R}} = \mathbf{0}$$

therefore

$$\mathbf{T}^{(0)}(\ln\mathbf{U}) = (\ln\mathbf{U})\mathbf{T}^{(0)}. \tag{55}$$

Formula (49) giving $\mathbf{T}^{(0)}$ in terms of the Cauchy stress and $\mathbf{V}$ can be substituted into eqn (55) to show that, because the principal axes of $\mathbf{V}$, $\mathbf{V}^2$, and $\ln\mathbf{V}$ coincide, eqn (55) is equivalent to

$$\mathbf{T}\mathbf{V} = \mathbf{V}\mathbf{T}. \tag{56}$$

Clearly, eqn (56) places restrictions on the constitutive equation which will be met only by special materials. Thus the logarithm of the left stretch does not, in general, have a conjugate stress.

However, in the case of an isotropic elastic material eqn (56) is met, and in this case the stress conjugate to $\ln\mathbf{V}$ is $III\,\mathbf{T}$ (see the discussion surrounding eqn (50)).

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